PLASTICITY (STRENGTH) CONDITION FOR A REINFORCED LAYER

Yu. V. Nemirovskii

A large number of studies, a summary of which may be found in [1-4], have been devoted to constructing strength and plasticity criteria for anisotropic materials. All these criteria have been formulated without regard to the nature of the anisotropy, and the admissibility of their use for particular classes of materials requires careful analysis.

The mechanical properties of the reinforced materials obviously depend on the properties of the bonding material and the reinforcing element materials, on the nature of the reinforcing and its percentage content. All these dependences must be reflected in one form or another in the plasticity conditions.

An approximate account for such relations can be made by introducing, as in [3, 4], a sufficient number of experimentally determined material constants into some hypothetical plasticity condition. We note that for any change of the composition materials, nature of the reinforcing, or its percentage content the need arises for new experiments to determine these constants.

Another approach is to construct a model of the reinforced material, accounting for its structure, properties of the elements, and peculiarities of their behavior, and then determine the plasticity condition for the material composition which is reflected by the adopted model.

This approach, used in the following, opens up the way to regulation of the reinforcing in order to improve the strength properties of the reinforced materials.

I. By reinforced layer we mean a comparatively thin plate consisting of an isotropic material with a reinforcing layer imbedded in it. The reinforcing layer is a network of fine, one-dimensional filaments arranged in directions forming the angles α_k ($k = 1, 2, \ldots, m$) with the direction which we hereafter denote by the subscript 1.

We assume that:

1. The material of all the elements comprising the reinforced layer will be rigid ideally-plastic and in the general case different for each element;

2. The number of reinforcing elements is sufficiently large that the reinforced layer can be considered quasiuniform;

3. In regard to composition joining we assume ideal adhesion, i.e., absence of slip between the binding and reinforcing elements;

4. The distance between the reinforcing elements is sufficiently large in comparison with their dimensions, and at the same time sufficiently small in comparison with the dimensions of the plate, that local effects near the filaments and irregularity of the deformation between two neighboring filaments can be neglected;

5. We postulate that each filament, if it belongs to the system of filaments embedded into the material, is capable of withstanding both tensile and compressive forces. However, since some instability mode may arise under the action of a compressive force, the elastic (strength) limits of the filaments in tension and compression are considered to be different.

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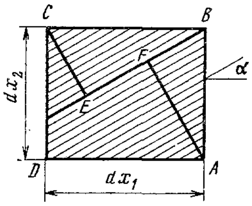

Fig. 1

Let ω_k be the density of the reinforcing filaments in the directions forming the angles α_k with the direction 1, let ω_z be the density of the reinforcing layer with respect to the plate thickness, h the thickness of the reinforced layer. We denote the principal directions of the orthogonal coordinate system in the plane of the plate by the subscripts 1 and 2. Then the internal forces in the composite layer are

$$\sigma_{ij} = a\sigma_{ij}^\circ + \sum_{k=1}^{m} \omega_k \sigma_k l_{ik} l_{jk}, \qquad i, j = 1, 2$$
$$\sigma_{ij} = T_{ij} / h, \quad l_{1k} = \cos\alpha_k, \quad l_{2k} = \sin\alpha_k, \quad 0 \leqslant \alpha_k \leqslant \pi \tag{1.1}$$
$$\omega_k = \frac{n_k F_k}{AFh}, \qquad a = 1 - \omega_z$$

Here T_{ij} are forces, σ_{ij}° are the stresses in the filler, σ_k are the stresses in the reinforcing filaments, F_k are the areas of the reinforcing element cross sections, n_k is the number of reinforcing element filaments with index k on the segment AF (Fig. 1).

For small deformations, on the basis of the assumption of absence of sliding we obtain the following relationships between the deformation rates ε_k of the angled reinforcement elements and the deformation rates ε_{ij} of the filler layer

$$\varepsilon_k = \varepsilon_{11} l_{1k}^2 + \varepsilon_{12} l_{1k} l_{2k} + \varepsilon_{22} l_{2k}^2 \tag{1.2}$$

Let us assume that the filler material is isotropic, has in the general case different properties in tension and compression, and obeys the Balandin plasticity condition [5]. Then for the plane stress state of an ideally rigid plastic body [5, 6] we have

$$\sigma_{11}^{\circ 2} - \sigma_{11}^\circ \sigma_{22}^\circ + \sigma_{22}^{\circ 2} + 3\sigma_{12}^{\circ 2} - (\sigma_0^- - \sigma_0^+) \ (\sigma_{11}^\circ + \sigma_{22}^\circ) - \sigma_0^- \sigma_0^+ = 0 \tag{1.3}$$

$$\varepsilon_{11} = \lambda(2\sigma_{11}^\circ - \sigma_{22}^\circ + \sigma_0^+ - \sigma_0^-), \quad \varepsilon_{22} = \lambda\,(2\sigma_{22}^\circ - \sigma_{11}^\circ + \sigma_0^+ - \sigma_0^-), \quad \varepsilon_{12} = 6\lambda\sigma_{12}^\circ \tag{1.4}$$

Here $\sigma_0^\pm$ are the yield limits of the binder in tension (plus) and compression (minus), λ is a positive multiplier.

Equation (1.3) will be satisfied identically if we take

$$\sigma_{11}^\circ + \sigma_0^+ - \sigma^-_0 = {}^2/_3 \sqrt{3}\sigma_0 \cos(\theta - {}^1/_6\pi) \sin\varphi, \quad \sigma_{12}^\circ = {}^1/_3 \sqrt{3}\sigma_0 \cos\varphi \tag{1.5}$$
$$\sigma_{22}^\circ + \sigma_0^+ - \sigma_0^- = {}^2/_3\sqrt{3}\,\sigma_0 \cos(\theta + {}^1/_6\pi) \sin\varphi$$
$$-\pi \leqslant \theta \leqslant \pi, \qquad 0 \leqslant \varphi \leqslant \pi, \qquad \sigma_0^2 = (\sigma_0^-)^2 + (\sigma_0^+)^2 - \sigma_0^+ \sigma_0^- \tag{1.6}$$

From (1.4) we have

$$\varepsilon_{11} = 2\lambda\sigma_0 \cos(\theta - {}^1/_3\pi) \sin\varphi, \quad \varepsilon_{22} = 2\lambda\sigma_0 \cos(\theta + {}^1/_3\pi) \sin\varphi, \quad \varepsilon_{12} = 2\sqrt{3}\,\lambda\sigma_0 \cos\varphi \tag{1.7}$$

We see from (1.6) and (1.7) that

$$\begin{array}{lll} \varepsilon_{11} > 0, & \varepsilon_{22} > 0 \ \text{ for } & -{}^1/_6\pi < \theta < {}^1/_6\pi \\ \varepsilon_{11} < 0, & \varepsilon_{22} < 0 \ \text{ for } & {}^5/_6\pi < \pm\theta < \pi \\ \varepsilon_{11} > 0, & \varepsilon_{22} < 0 \ \text{ for } & {}^1/_6\pi < \theta < {}^5/_6\pi \\ \varepsilon_{11} < 0, & \varepsilon_{22} > 0 \ \text{ for } & -{}^5/_6\pi < \theta < -{}^1/_6\pi \end{array} \tag{1.8}$$
$$\varepsilon_{12} \geqslant 0 \ \text{ for } \ 0 \leqslant \varphi \leqslant {}^1/_2\pi, \qquad \varepsilon_{12} \leqslant 0 \ \text{ for } \ {}^1/_2\pi \leqslant \varphi \leqslant \pi \tag{1.9}$$

From (1.1) and (1.3), considering the conditions for uniaxiality of the stress-strain state in the reinforcing elements and the possibilities for the signs of the strains in these elements in accordance with (1.8), (1.9), (1.2), in the general case we obtain the limiting relations

$$(\sigma_{11} - k_{11})^2 - (\sigma_{11} - k_{11})(\sigma_{22} - k_{22}) + (\sigma_{22} - k_{22})^2 + 3(\sigma_{12} - k_{22})^2 = (a\,\sigma_0)^2$$
$$k_{11} = a(\sigma_0^- - \sigma_0^+) + \sum_{k=1}^{m} \omega_k a_k^\pm \sigma_k^\pm l_{1k}^2, \qquad k_{12} = \sum_{k=1}^{m} \omega_k a_k^\pm \sigma_k^\pm l_{1k} l_{2k} \tag{1.10}$$
$$k_{22} = a(\sigma_0^- - \sigma_0^+) + \sum_{k=1}^{m} \omega_k a_k^\pm \sigma_k^\pm l_{2k}^2, \qquad a_k^+ = 1, \qquad a_k^- = -1$$

Here $\sigma_k^\pm$ are the yield limits of the reinforcing elements in tension (+) and compression (-).

The question of the choice of the upper signs for $\sigma_k^{\pm}$ is resolved in accordance with the signs of the strains ε_k. The regions of choice of particular signs as a function of the parameters θ and φ for given values of the reinforcement angles α_k can be established using (1.2), (1.8), (1.9).

Since in the general case the quantities $a_k^{\pm}$ are independent of one another, (1.10) in the σ_{11}, σ_{22}, σ_{12} stress space defines 2^m ellipsoids, which are obtained by a parallel shift of the ellipsoid (1.3).

In addition to the limiting relations (1.10), which correspond to full utilization of the carrying capability of all the reinforcing elements and the filler layer, several limiting relations are possible which correspond to the stiffness conditions of the reinforcing elements of any direction.

Assume the reinforcing elements which form the angles α_k with the axis 1 remain rigid. In this case the stress σ_k remains undefined and the strain rate $\varepsilon_k = 0$, Consequently, from (1.2) and (1.7)

$$[l_{1k}^2 \cos(\theta - {}^1/_3\,\pi) + l_{2k}^2 \cos(\theta + {}^1/_3\,\pi)] \sin\varphi + \sqrt{3}\, l_{1k} l_{2k} \cos\varphi = 0 \tag{1.11}$$

For a given value of the reinforcement angle α_k this equation makes it possible to find φ in terms of θ. Then (1.1) define the three quantities σ_{11}, σ_{22}, σ_{12} in terms of the two parameters θ and σ_k and, consequently, yield the final limiting relation between the quantities σ_{11}, σ_{22}, σ_{12}. We shall omit here the further calculations leading to exclusion of the parameters θ and σ_k. It is not difficult to recover these expressions if necessary. The corresponding operations will be demonstrated below using particular examples. In so doing we must bear in mind that in (1.1) we must set $\sigma_n = a_n^{\pm}\sigma_n^{\pm}$ for all θ and φ for which $a_n \pm \varepsilon_n > 0$, i.e.,

$$\begin{gathered} a_n^{\pm}\{[l_{1n}^2\cos(\theta - {}^1/_3\pi) + l_{2n}^2\cos(\theta + {}^1/_3\pi)]\sin\varphi + \sqrt{3}\, l_{1n} l_{2n}\cos\varphi\} > 0, \\ a_n^{\pm} = \pm 1,\; n = 1, 2, \ldots, m, \quad n \neq k \end{gathered} \tag{1.12}$$

In the σ_{11}, σ_{12}, σ_{22} space the limiting relations corresponding to this case define cylindrical surfaces with generators parallel to the direction of the rigid filaments.

Finally, the case is possible in which some two families of the reinforcing filaments remain rigid. Equating in this case ε_k to zero for two concrete values of k (for example 1 and 2) and using (1.2) and (1.7) we define the parameters θ and φ. Then the quantities σ_{ij}°, ε_k, and σ_k (for $k > 2$) become known, while the stresses σ_1 and σ_2 in the rigid filaments remain undetermined. Then, excluding σ_1 and σ_2 from (1.1) we obtain a linear relation between the quantities σ_{ij}, which defines a plane in the σ_{11}, σ_{12}, σ_{22} space. Moreover the strain rate vector with the components ε_{ij} will be orthogonal to this plane.

We see from (1.3) and (1.4) that for an isotropic binder the limit surface in stress space has the form of a convex ellipsoid and the associated strain law corresponds to the condition of orthogonality of the strain rate vector with the components ε_{11}, ε_{22}, ε_{12} to the elliposid at a given point on the surface.

For the reinforced layer the combined limit surface, being convex, consists in the general case of a large number of pieces of different analytic surfaces, and the strain law, as we see from the construction of the limit surface, again corresponds to the condition of orthogonality of the strain rate vector with the components ε_{11}, ε_{12}, ε_{22} to this surface on each of its pieces.

II. Let us examine the particular case in which

$$m = 4,\ \sigma_{12} = 0,\ \alpha_1 = 0,\ \alpha_2 = \pi/2,\ \omega_3 = \omega_4 = \omega,\ \alpha_3 = \alpha,\ \alpha_4 = \pi - \alpha$$

Then

$$\omega_3\sigma_3 l_{13} l_{23} = -\,\omega_4\sigma_4 l_{14} l_{24},\quad \sigma_3 = \sigma_4 = \sigma,\quad k_{12} = 0,\quad \sigma_{12}^\circ = 0,\quad \varphi = \pi/2$$

The angled reinforcement element stiffness condition (1.11) takes the form

$$\cos^2\alpha\,\cos(\theta - {}^1/_3\,\pi) + \sin^2\alpha\,\cos(\theta + {}^1/_3\,\pi) = 0 \tag{2.1}$$

This equality is possible only for ${}^1/_6\,\pi < \theta < {}^5/_6\,\pi$ or $-{}^5/_6\,\pi < \theta < -{}^1/_6\,\pi$. But in the first case $\varepsilon_{11} > 0$, $\varepsilon_{22} < 0$, and $\sigma_1 = -\sigma_1^+$, $\sigma_2 = -\sigma_2^-$, while in the second case $\varepsilon_{11} < 0$, $\varepsilon_{22} > 0$, and $\sigma_1 = -\sigma_1^-$, $\sigma_2 = \sigma_2^+$.

Substituting these values together with (1.5) into the first two equalities (1.1) and excluding $\sigma = \sigma_3 = \sigma_4$ and θ with the aid of (2.1), we obtain

$$\begin{gathered} [\sigma_{22} + a(\sigma_0^+ - \sigma_0^-) \pm \omega_2\sigma_2^{\mp}]\cos^2\alpha - [\sigma_{11} + a(\sigma_0^+ - \sigma_0^-) \\ \mp \omega_1\sigma_1^{\pm}]\sin^2\alpha = \mp\, {}^1/_6\sqrt{3}\,a\sigma_0\,(1 + 3\cos^2 2\alpha) \end{gathered} \tag{2.2}$$

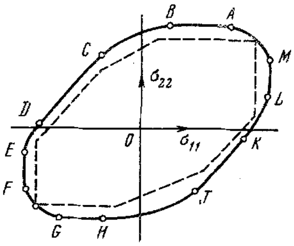

Fig. 2

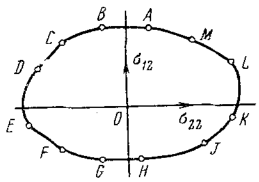

Fig. 3

where the upper signs correspond to values of the parameter θ from the interval $^1/_6\,\pi < \theta < {}^5/_6\pi$; the lower values are for those from the interval $-{}^5/_6\,\pi < \theta < -{}^1/_6\,\pi$.

In the case of reinforcing elements which are stiff in the directions 1 and 2, we have similarly

$$\sigma_{ii} = \mp\,\omega_i\sigma_i^{\mp} + a\,(\sigma_0^{r} - \sigma_0^{+} \mp {}^2/_3\sqrt{3}\,\sigma_0)\;(i = 1,2) \tag{2.3}$$

Thus, in the particular case in question the limit curve in the σ_{11}, σ_{22} plane is defined by (1.10), (2.2), (2.3).

The form of this curve is shown schematically by the continuous curve in Fig. 2. Here the arcs MA, BC, DE, FG, HJ, KL are defined by (1.10), the straight lines AB and GH, ML, and EF are defined by (2.3), and the straight lines CD and JK are defined by (2.2). If in a particular case the Tresca condition [5] is used as the basic plasticity condition for the isotropic material of the filler layer, the limit curve for the reinforced layer will have the octagon form shown schematically in Fig. 2 by the dashed line [6].

In the absence of reinforcing elements in a particular direction, the corresponding pairs of straight lines in Fig. 2 degenerate into points, and the arcs connected by them merge when extended. Thus, in the absence of the angled reinforcement elements ($\omega = 0$) the straight lines DC and JK disappear; for $\omega_1 = 0$ the straight lines AB and GH disappear. The suggested approach makes it possible to indicate the region of variation of the reinforcement parameters and loads for which a particular portion of the limit curve or surface is realized, and thereby makes it possible to regulate the nature of the structure reinforcement for the given loading conditions. Thus, for example, for the straight line AB in Fig. 2 we have for $\sigma_0^+ = \sigma_0^- = \sigma_0$

$$\varepsilon_{11} = 0, \quad \varepsilon_{22} = 0, \quad \varepsilon_3 = \varepsilon_4 > 0,$$

$$\sigma_{11}^{\circ} = {}^1/_3\sqrt{3}\sigma_0, \quad \sigma_3 = \sigma_4 = \sigma^+, \quad -\sigma_1^- \leqslant \sigma_1 \leqslant \sigma_1^+$$

So that

$$-\omega_1\sigma_1^- \leqslant \sigma_{11} - {}^1/_3\sqrt{3}\sigma_0\,a - 2\omega\sigma^+\cos^2\alpha \leqslant \omega_1\sigma_1^+$$

For a given value of σ_{11} this inequality defines the relationship between the reinforcement parameters for which the longitudinal structure does not exhaust the load carrying capability.

III. Let us examine as the second example the particular case $m = 4$, $\alpha_1 = 0$, $\alpha_2 = {}^1/_2\pi$, $\sigma_0^+ = \sigma_0^- = \sigma_0$, $\sigma_{11} = 0$, $\varepsilon_{11} = 0$, which corresponds to the presence of rigid reinforcement in direction 1 in the absence of the corresponding load. In this case part of the limit relations are defined by the equalities

$$(\sigma_{22} - k_{22})^2 + 4\,(\sigma_{12} - k_{12})^2 = {}^4/_3\,(a\sigma_0)^2 \tag{3.1}$$

while the remaining relations correspond to the cases $\varepsilon_3 = 0$, $\varepsilon_4 = 0$, $\varepsilon_{22} = 0$.

Let us examine these cases in more detail. We have for $\sigma_{11} = 0$, $\varepsilon_{11} = 0$

$$\varepsilon_3 = \varepsilon_{22}\sin^2\alpha_3\,(1 - 2\operatorname{ctg}\varphi\operatorname{ctg}\alpha_3), \quad \varepsilon_4 = \varepsilon_{22}\sin^2\alpha_4\,(1 - 2\operatorname{ctg}\varphi\operatorname{ctg}\alpha_4) \tag{3.2}$$

$$\theta = \frac{\pi\,(2 \pm 3)}{6}, \quad \sigma_{11}^{\circ} = -\frac{1}{2}\sigma_{22}^{\circ} = \pm\frac{\sqrt{3}\sigma_0}{3}\sin\varphi, \quad \sigma_{12}^{\circ} = \frac{\sqrt{3}\sigma_0}{3}\cos\varphi \tag{3.3}$$

Let $\theta = {}^5/_6\pi$. Then $\varepsilon_{22} < 0$, $\sigma_2 = -\sigma_2^-$. If in this case $\varepsilon_3 = 0$, we obtain from (3.2)

$$\varphi = \varphi_1 = {}^1/_2\pi - \operatorname{arc\,tg}({}^1/_2\operatorname{tg}\alpha_3), \quad \varepsilon_4 = \varepsilon_{22}\sin^2\alpha_4\,(1 - 2\operatorname{ctg}\varphi_1\operatorname{ctg}\alpha_4) \tag{3.4}$$

For a known value of α_3 the sign of ε_4 is known, and then the stress σ_4 is equal to the yield limit in tension or compression.

Considering this situation and substituting (3.3) for $\varphi = \varphi_1$ into (1.1), after excluding σ_3 we obtain the limit relation

$$2\sqrt{3}\,(\sigma_{12}\sin\alpha_3 - \sigma_{22}\cos\alpha_3) = 2\,(2\sigma_0\sin\varphi_1 + \sqrt{3}\omega_2\sigma_2^- - \sqrt{3}\omega_4\sigma_4\sin^2\alpha_4)\cos\alpha_3 + (2\sigma_0\cos\varphi_1 + \sqrt{3}\omega_4\sigma_4\sin 2\alpha_4)\sin\alpha_3 \tag{3.5}$$

Similarly, in the case $\varepsilon_4 = 0$ we obtain

$$2\sqrt{3}\,(\sigma_{12}\sin\alpha_4 - \sigma_{22}\cos\alpha_4) = 2\,(2\sigma_0\sin\varphi_2 + \sqrt{3}\omega_2\sigma_2^- - \sqrt{3}\omega_3\sigma_3\sin^2\alpha_3)\cos\alpha_4 + (2\sigma_0\cos\varphi_2 + \sqrt{3}\omega_3\sigma_3\sin 2\alpha_3)\sin\alpha_4 \tag{3.6}$$

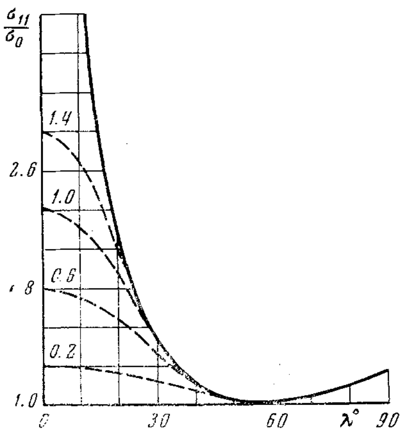

Fig. 4

Here φ_2 equals

$$\varphi_2 = {}^1/_2\,\pi - \operatorname{arc\,tg}\,({}^1/_2\,\operatorname{tg}\alpha_4)$$

and σ_3 equals the yield limit in tension or compression, depending on the sign of the quantity

$$\varepsilon_3 = \varepsilon_{22}\sin^2\alpha_3\,(1 - 2\operatorname{ctg}\varphi_2\operatorname{ctg}\alpha_3) \tag{3.7}$$

In the same fashion we obtain for $\theta = {}^1/_6\,\pi$ the two limit relations

$$2\sqrt{3}\,(\sigma_{22}\cos\alpha_3 - \sigma_{12}\sin\alpha_3) = 2\,(2\sigma_0\sin\varphi_1 + \sqrt{3}\omega_2\sigma_2^+ + \sqrt{3}\omega_4\sigma_4\sin 2\alpha_4)\sin\alpha_3 + \sqrt{3}\omega_4\sigma_4\sin^2\alpha_4)\cos\alpha_3 - (2\sigma_0\cos\varphi_1 + \sqrt{3}\omega_4\sigma_4\sin 2\alpha_4)\sin\alpha_3 \tag{3.8}$$

$$2\sqrt{3}\,(\sigma_{22}\cos\alpha_4 - \sigma_{12}\sin\alpha_4) = 2\,(2\sigma_0\sin\varphi_2 + \sqrt{3}\omega_2\sigma_2^+ + \sqrt{3}\omega_4\sigma_4\sin^2\alpha_3)\cos\alpha_4 - (2\sigma_0\cos\varphi_2 + \sqrt{3}\omega_3\sigma_3\sin 2\alpha_3)\sin\alpha_4 \tag{3.9}$$

Here σ_3, σ_4 are also equal to the yield limits in tension or compression, depending on the signs of the quantities ε_3, ε_4, defined by (3.4) and (3.7). Here we must bear in mind that in the present case $\varepsilon_{22} > 0$.

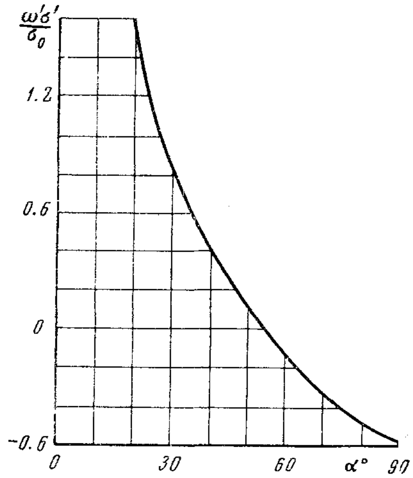

Fig. 5

In the case $\varepsilon_{22} = 0$, $\varepsilon_{11} = 0$ we have

$$\varphi = 0,\ \pi,\ \sigma_{11}^\circ = \sigma_{22}^\circ = \sigma_0^- - \sigma_0^+,\ \sigma_{12}^\circ = \pm{}^1/_3\sqrt{3}\sigma_0, \varepsilon_3 = {}^1/_2\sin 2\alpha_3,\ \varepsilon_4 = {}^1/_2\sin 2\alpha_4$$

and the stresses σ_1, σ_2 are indeterminate. For a known value of the reinforcement angles α_3, α_4 the signs of ε_3 and ε_4, and therefore the quantities σ_3, σ_4 as well, are known in the limit state. And in this case the limit state

$$\sigma_{12} = \pm{}^1/_3\sqrt{3}a\sigma_0 + {}^1/_2\,(\omega_3\sigma_3\sin 2\alpha_3 + \omega_4\sigma_4\sin 2\alpha_4) \tag{3.10}$$

Here σ_3 and σ_4 are equal to the yield limits in tension and compression, depending on the signs of ε_3 and ε_4.

The form of the limit curve in the σ_{12}, σ_{22} plane for the particular case in question is shown schematically in Fig. 3. The straight lines AB and GH are defined by (3.10), the straight lines CD, EF, JK, LM are defined by (3.5), (3.6), (3.8), (3.9), and the curves MA, BC, DE, FG, HJ, KL are defined by (3.1).

Here, for each segment of the limit curve we can indicate the corresponding region of variation of the loads and reinforcement parameters. For example, let us examine the horizontal straight line segment AB in Fig. 3 in the case $0 \le \alpha_1 \le {}^1/_2\pi$, ${}^1/_2\,\pi \le \alpha_2 \le \pi$. Then it is not difficult to see that $\sigma_{11}^\circ = 0$, $\sigma_3 = \sigma_3^+$, $\sigma_4 = -\sigma_4^-$, $\sigma_{22}^\circ = 0$, $-\sigma_1^- \le \sigma_1 \le \sigma_1^+$, $-\sigma_2^- \le \sigma_2 \le \sigma_2^+$ and, consequently,

$$-\omega_1\sigma_1^- \leqslant \omega_3\sigma_3^+\cos^2\alpha_1 - \omega_4\sigma_4^-\cos^2\alpha_2 \leqslant \omega_1\sigma_1^+$$
$$-\omega_2\sigma_2^- \leqslant \sigma_{22} - \omega_3\sigma_3^+\sin^2\alpha_1 + \omega_4\sigma_4^-\sin^2\alpha_2 \leqslant \omega_2\sigma_2^+$$

If, in addition, we take

$$\alpha_1 = \alpha,\ \alpha_2 = \pi - \alpha,\ \omega_3 = \omega_4 = \omega$$
$$\sigma_3^+ = k\sigma_3^- = \sigma_4^+ = k\sigma_4^- = \sigma_1^+ = k\sigma_1^- = \sigma_2^+ = k\sigma_2^- \quad (k \leqslant 1)$$

we obtain

$$-\omega_1 k \leqslant \omega\,(1 - k)\cos^2\alpha \leqslant \omega_1$$
$$-\omega_2\sigma_2^+k \leqslant \sigma_{22} - \omega_2\sigma_2^+\,(1 - k)\sin^2\alpha \leqslant \omega_2\sigma_2^+$$

Similar inequalities can be obtained for each of the segments of the limit curve in Fig. 3.

Let us examine the case of a unidirectional reinforced material which is stretched by a force which forms the angle α with the reinforcement direction. Then, setting in (1.1) $\omega_n = 0$ ($n = 2, \ldots, m$), $\omega_1 = \omega$, $\sigma_1 = \sigma$, $\sigma_{12} = \sigma_{22} = 0$, $\alpha_1 = \alpha$ and considering (1.5), we obtain for $\sigma_0^+ = \sigma_0^- = \sigma_0$

$$\sigma_{11} = {}^2/_3\sqrt{3}\,\sigma_0\,a\cos(\theta - {}^1/_6\,\pi)\sin\varphi + \omega\sigma\cos^2\alpha \tag{4.1}$$

$${}^2/_3\sqrt{3}\,a\sigma_0 a\cos(\theta + {}^1/_6\,\pi)\sin\varphi + \omega\sigma\sin^2\alpha = 0,\quad {}^1/_3\sqrt{3}a\,\sigma_0\cos\varphi + {}^1/_2\,\omega\sigma\sin^2\alpha = 0 \tag{4.2}$$

Excluding $\omega\sigma$ from (4.2), we obtain ($\alpha \ne {}^1/_2\pi$, $\varphi \ne {}^1/_2\,\pi$)

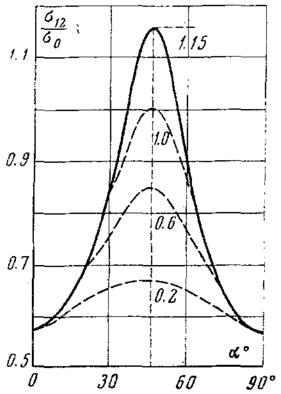

Fig. 6

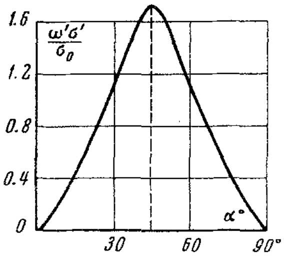

Fig. 7

$$2\cos(\theta + {}^1/_6\pi) = \operatorname{ctg}\varphi \operatorname{tg}\alpha \qquad (4.3)$$

If the reinforcing elements remain rigid, from the condition $\varepsilon_1 = 0$ with account for (1.2) and (1.7) we obtain

$$\sqrt{3}\operatorname{tg}\theta = 1 + 6\cos^2\alpha \qquad (4.4)$$

Equations (4.1), (4.3), (4.4) together define the limit tensile force of the reinforced layer in the case in which the reinforcing elements remain rigid. The curve of $\sigma_{11}/a\sigma_0$ versus α calculated from these equations is shown by the continuous curve in Fig. 4. With the aid of (4.2), (4.3), (4.4) we can now calculate as a function of α the magnitude of the forces which arise in this case in the reinforcing elements. The corresponding curve is shown in Fig. 5. This curve defines the maximal rational strength of the reinforcing elements. Further increase of their strength does not lead to increase of the material strength. Therefore the continuous curve in Fig. 4 will be the theoretical material strength. We note that in (4.3) and (4.4) it was assumed that $\alpha \neq {}^1/_2\pi$, $\varphi \neq {}^1/_2\pi$. If $\alpha = {}^1/_2\pi$, $\varphi = {}^1/_2\pi$, then it is easy to see that we have

$$\sigma_{11} = \pm {}^2/_3\sqrt{3}\sigma_0 a, \qquad \omega\sigma = \mp\sqrt{3}\sigma_0 a$$

If for given reinforcement density and angle the yield (strength) limit of the reinforcing element material is such that the quantity $\omega\sigma^{\pm}/a\sigma_0$ lies below the curve in Fig. 5, then exhaustion of the load carrying capability of the reinforced layer is accompanied by exhaustion of the load carrying capability of the reinforcing elements. In this case we obtain the limit load for a given value of $\omega\sigma/a\sigma_0$ with the aid of (4.1), (4.3), and the second equation (4.2). The corresponding curves for $\omega\sigma/a\sigma_0 = 0.2, 0.6, 1.0, 1.4$ are shown dashed in Fig. 4.

It may be shown similarly that the shear yield limit in the case of rigid reinforcing elements is defined by the equality

$$\frac{\sigma_{12}}{a\sigma_0} = \frac{4\sin\varphi\cos\theta}{3\sin 2\alpha}$$

In this case the quantities φ, θ, and $\omega\sigma/a\sigma_0$ are found from the equalities

$$\operatorname{tg}\theta = \sqrt{3}\cos 2\alpha, \qquad \operatorname{ctg}\varphi = -\frac{(1 + 3\cos^2 2\alpha)\cos\theta}{\sqrt{3}\sin^2\alpha}, \quad \omega\sigma / a\sigma_0 = -2\sin\varphi\cos\theta$$

The corresponding theoretical strength curves are shown by the continuous curves in Figs. 6 and 7.

However, if exhaustion of the load carrying capability of the reinforced layer is accompanied by exhaustion of the load carrying capability of the reinforcing elements, then the yield (strength) limit for the given values of $\omega\sigma/a\sigma_0$ is defined by the equality

$$\sigma_{12} / a\sigma_0 = {}^1/_3\sqrt{3}\cos\varphi + \omega\sigma / a\sigma_0 \sin 2\alpha$$

where φ is found from the equality

$$\varphi = -\arcsin \omega\sigma / 2\sigma_0 a\cos\theta, \; \theta = \operatorname{arc\,tg}(\sqrt{3}\cos 2\alpha)$$

The corresponding curves for $\omega\sigma/a\sigma_0 = 0.2, 0.6, 1.0$ are shown dashed in Fig. 6.

It was assumed above that $\alpha \neq 0, {}^1/_2\pi, \pi$. However, if $\alpha = 0, {}^1/_2\pi, \pi$, it is easy to see that

$$\sigma_{12} = \pm\sqrt{3}\sigma_0 a, \qquad \omega\sigma = 0$$

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